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Zero Locus of a Beam with Varying Sensor and Actuator Locations

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The nonrational transfer function of an important component of a flexible structure, the beam, is analyzed. The true pattern of zeros of that transfer function is investigated as a function of sensor and actuator separation. Translational displacement sensors are used for two cases in which a force input and a moment input are separately applied at one end of the beam. Contrary to general opinion, zeros do not simply approach the origin along the real axis of the *s* plane as the sensor is moved away from the point of force/moment application. When the displacement sensor is located at a certain point, the first pair of zeros on the real axis of the *s* plane arrive at the origin and cancel the rigid-body mode. The location of the translational displacement sensor on the beam at which the rigid-body mode of the beam is unobservable is analyzed as the *center of percussion* and is uniquely located for each case. If the sensor is moved beyond such a point, a pair of zeros appear on the imaginary axis and move away from the origin along the imaginary axis of the *s* plane. Finally, the zero migration associated with an angular displacement sensor is briefly considered and compared with the zero migration associated with the translational displacement sensor.

I. Introduction

THE major problem associated with modeling a distributed parameter system is that a mathematical model which is described by a partial differential equation is never completely accurate and almost impossible to solve in practice. A flexible structure may be approximated as a multimass system, or a lumped-parameter system,[1-3] using the finite element method, which avoids these modeling difficulties as well as solving the governing partial differential equations. Such an approach provides a reasonably accurate representation of modal frequencies. For these reasons, the multimass model is generally preferred for studies of flexible structures.[4-6] However, if the masses are cascaded or otherwise arranged to eliminate interconnections between nonadjacent elements, as is generally the case, the resulting system is always minimum phase no matter where the actuators or sensors are located. Furthermore, for the collocated case, the resulting transfer function has alternating poles and zeros on the imaginary axis.[7] It is shown in Ref. 7 that when the poles and zeros alternate on the imaginary axis, the system can always be stabilized with direct velocity feedback alone.

However, the dynamic motion of many controlled flexible structures can be characterized in terms of the poles and zeros of transcendental transfer functions (nonrational transfer functions), which are not approximations. The poles of such a transfer function correspond to the modal frequencies of the system, but the zero values of the function are determined by the locations of actuators and sensors in the system. An actual flexible structure, e.g., a flexible beam, can only be described by a transcendental transfer function, and it always has *nonminimum* phase zeros when the sensors and actuators are noncollocated.[8] This means that although the multimass-spring model for a flexible structure can be used for a collocated case, it is a poor model for representing a noncollocated system as it inadequately predicts the presence and location of the system zeros. Therefore, a multimass-spring model is useful for determining only the location of the poles, and not the zeros for a noncollocated system.

Since a beam is a basic component of a flexible structure, this paper investigates the true pattern of zeros for a beam as a function of the sensor and actuator separation. For convenience, the actuator input is applied at one end of the beam, and the sensor is located on the beam at an arbitrary distance from the input.

In Sec. II two different physical arrangements are considered. First, a force input is applied at one end of the beam, and a translational displacement sensor is used. Second, with the same sensor configuration, a moment input is applied at the same beam end. The migration of the zeros with respect to sensor location for both of the complex transfer functions is explained, and some comments are made about the unobservability of the rigid-body mode when the sensor is placed at the *center of percussion* of the beam in Sec. III. In Sec. IV concluding remarks are given.

II. Transfer Functions of a Beam with a Force Input and a Moment Input

Two cases are investigated. A force input is applied at the left end of the beam for the first case, and this same beam end

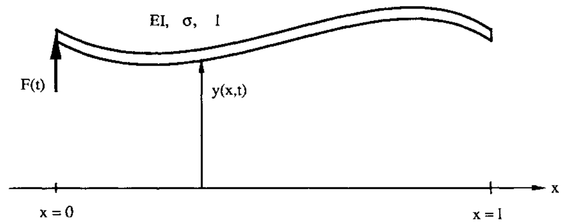

Fig. 1 Uniform Bernoulli-Euler beam with input force at the left end.

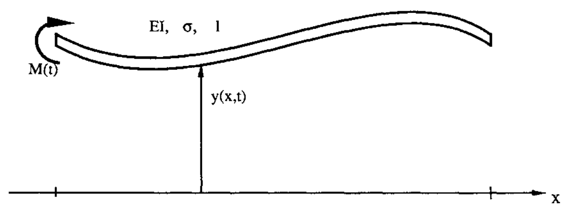

Fig. 2 Uniform Bernoulli-Euler beam with input moment at the left end.

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is subjected to a moment input in the second case. In both cases, the translational displacement is measured.

A beam structure conforming to the Bernoulli-Euler model, neglecting shear distortion and rotary inertia, exhibits transversal motion when a force or a moment input is applied at the left end of the beam as shown in Figs. 1 and 2. The equation of motion can be written in dimensionless form as follows:

$$y''''(x,t) + \ddot{y}(x,t) = 0 \tag{1}$$

with boundary conditions for a force input

$$y''(0,t) = y''(1,t) = 0$$

$$y'''(0,t) = F(t)$$

$$y'''(1,t) = 0$$

and separate boundary conditions for a moment input

$$y''(0,t) = M(t)$$

$$y''(1,t) = 0$$

$$y'''(0,t) = y'''(1,t) = 0$$

where

$y(x,t)$ = transversal displacement
x,y = units of length
$F(t)$ = control input force with units of EI/l^2 applied at the left end, $x=0$
$M(t)$ = control input moment with unit of EI/l applied at the left end, $x=0$
l = length of the beam
t = time with units of $(\sigma l^4/EI)^{1/2}$
EI = bending stiffness
σ = mass per unit length
y'''' = $\partial^4 y/\partial x^4$
$\ddot{y}$ = $\partial^2 y/\partial t^2$

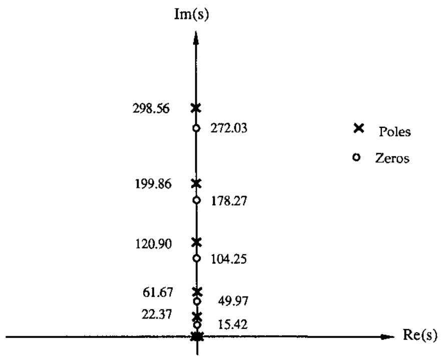

Fig. 3 Poles and zeros of the force input transfer function for a collocated case where $x = 0.0$.

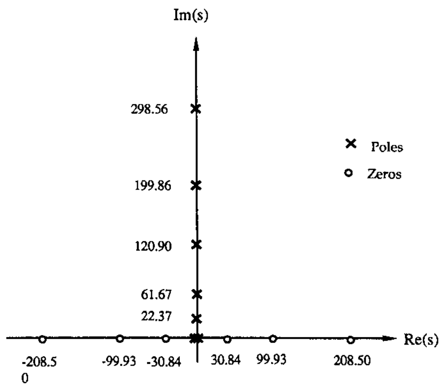

Fig. 4 Poles and zeros of the force input transfer function for a noncollocated case when $x = 1.0$.

Taking the Laplace transform of Eq. (1) with zero initial conditions with respect to t results in

$$\bar{y}''''(x,s) - \lambda^4 \bar{y}(x,s) = 0 \tag{2}$$

subject to

$$\bar{y}''(0,s) = \bar{y}''(1,s) = 0$$

$$\bar{y}'''(0,s) = \bar{F}(s)$$

$$\bar{y}'''(1,s) = 0 \tag{3}$$

for a force input case, and

$$\bar{y}''(0,s) = \bar{M}(s)$$

$$\bar{y}''(1,s) = 0$$

$$\bar{y}'''(0,s) = \bar{y}'''(1,s) = 0 \tag{4}$$

for a moment input case, where

$\lambda^4 = -s^2$
$\bar{y}(x,s)$ = Laplace transformation of $y(x,t)$
$\bar{F}(s)$ = Laplace transformation of $F(t)$
$\bar{M}(s)$ = Laplace transform of $M(t)$

The general solution[9] of Eq. (2) is

$$\bar{y}(x,s) = A_1 \sin \lambda x + A_2 \cos \lambda x + A_3 \sinh \lambda x + A_4 \cosh \lambda x \tag{5}$$

By applying the force input boundary conditions of Eq. (3) to the general solution of Eq. (5), the coefficients A_1, A_2, A_3, and A_4 can be obtained as follows:

$$A_1 = \frac{\bar{F}(s)}{\lambda^3 \Delta(\lambda)} (1 - \sin\lambda \cdot \sinh\lambda - \cos\lambda \cdot \cosh\lambda)$$

$$A_2 = \frac{\bar{F}(s)}{\lambda^3 \Delta(\lambda)} (\cosh\lambda \cdot \sin\lambda - \sinh\lambda \cdot \cos\lambda)$$

$$A_3 = \frac{\bar{F}(s)}{\lambda^3 \Delta(\lambda)} (\cos\lambda \cdot \cosh\lambda - \sin\lambda \cdot \sinh\lambda - 1)$$

$$A_4 = A_2$$

where

$$\Delta(\lambda) = 2\lambda^3 \cdot (1 - \cos\lambda \cdot \cosh\lambda)$$

Then, the transcendental transfer function from the control input force $\bar{F}(s)$ to the transversal displacement $\bar{y}(x,s)$ at the location of the sensor x is represented as

$$\begin{aligned} G(x,s) &= \frac{\bar{y}(x,s)}{\bar{F}(s)} \\ &= \frac{1}{\Delta(\lambda)} \big[(\sinh\lambda \cdot \cos\lambda - \cosh\lambda \cdot \sin\lambda)(\cosh\lambda x + \cos\lambda x) \\ &\quad + \sinh\lambda \cdot \sin\lambda (\sinh\lambda x + \sin\lambda x) \\ &\quad + (1 - \cos\lambda \cdot \cosh\lambda)(\sinh\lambda x - \sin\lambda x) \big] \end{aligned} \tag{6}$$

For the collocated case ($x = 0$)

$$G(0,s) = \frac{\bar{y}(0,s)}{\bar{F}(s)} = \frac{\sinh\lambda \cdot \cos\lambda - \cosh\lambda \cdot \sin\lambda}{\lambda^3 (1 - \cos\lambda \cdot \cosh\lambda)} \tag{7}$$

and for the particular noncollocated case, where a sensor is at the right end ($x = 1$)

$$G(1,s) = \frac{\bar{y}(1,s)}{\bar{F}(s)} = \frac{\sinh\lambda - \sin\lambda}{\lambda^3 (1 - \cos\lambda \cdot \cosh\lambda)} \tag{8}$$

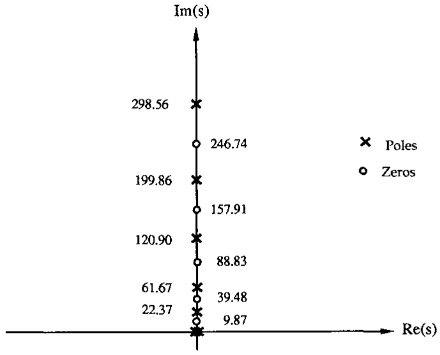

Fig. 5 Poles and zeros of the moment input transfer function for a collocated case when $x = 0.0$.

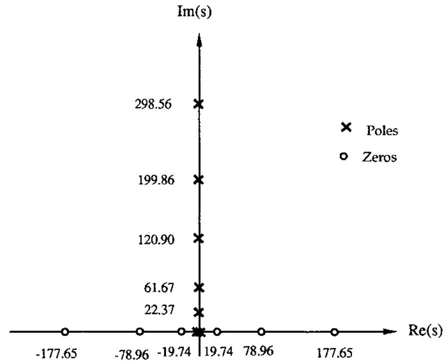

Fig. 6 Poles and zeros of the moment input transfer function for a noncollocated case when $x = 1.0$.

The transfer function $G(0,s) = \bar{y}(0,s)/\bar{F}(s)$ has alternating poles and zeros on the imaginary axis of the s plane,[7] as shown in Fig. 3, and is *minimum phase*. However, the transfer function $G(1,s) = \bar{y}(1,s)/\bar{F}(s)$ has an infinite number of zeros on the real axis of the s plane, as shown in Fig. 4, and is *nonminimum phase*. Note that for any noncollocated case, the exact representation of a nondispersive system such as a beam can have a nonminimum phase transfer function with real zeros (no complex zeros), while the exact representation of a dispersive system such as a bar structure, which can only deform along the longitudinal axis, has a minimum phase transfer function.[8]

Similarly, for a moment input case, by applying the boundary conditions of Eq. (4), the coefficients A_1, A_2, A_3, and A_4 of the general solution in Eq. (5) are obtained as follows:

$$A_1 = \frac{\bar{M}(s)}{\lambda^2 \Delta(\lambda)} (\cosh\lambda \cdot \sin\lambda + \sinh\lambda \cdot \cos\lambda)$$

$$A_2 = \frac{\bar{M}(s)}{\lambda^2 \Delta(\lambda)} (\cosh\lambda \cdot \cos\lambda - \sinh\lambda \cdot \sin\lambda - 1)$$

$$A_3 = A_1$$

$$A_4 = \frac{\bar{M}(s)}{\lambda^2 \Delta(\lambda)} (1 - \cosh\lambda \cdot \cos\lambda - \sinh\lambda \cdot \sin\lambda)$$

where

$$\Delta(\lambda) = 2\lambda^2 \cdot (1 - \cos\lambda \cdot \cosh\lambda)$$

Then, the transcendental transfer function from the control input moment $\bar{M}(s)$ to the transversal displacement $\bar{y}(x,s)$ at the location of the sensor x is represented as

$$H(x,s) = \frac{\bar{y}(x,s)}{\bar{M}(s)}$$

$$= \frac{1}{\Delta(\lambda)} \big[(\cosh\lambda \cdot \sin\lambda + \sinh\lambda \cdot \cos\lambda)(\sin\lambda x + \sinh\lambda x)$$

$$- \sinh\lambda \cdot \sin\lambda(\cosh\lambda x + \cos\lambda x)$$

$$+ (1 - \cosh\lambda \cdot \cos\lambda)(\cosh\lambda x - \cos\lambda x)\big] \tag{9}$$

For the collocated case ($x = 0$)

$$H(0,s) = \frac{\bar{y}(0,s)}{\bar{M}(s)} = -\frac{\sinh\lambda \cdot \sin\lambda}{\lambda^2(1 - \cos\lambda \cdot \cosh\lambda)} \tag{10}$$

and for a noncollocated case ($x = 1$)

$$H(1,s) = \frac{\bar{y}(1,s)}{\bar{M}(s)} = \frac{\cosh\lambda - \cos\lambda}{\lambda^2(1 - \cos\lambda \cdot \cosh\lambda)} \tag{11}$$

Once again, $\lambda^4 = -s^2$, and the transfer function $H(0,s) = \bar{y}(0,s)/\bar{M}(s)$ is minimum phase with alternating poles and zeros on the imaginary axis of the s plane,[7] as shown in Fig. 5. The transfer function $H(1,s) = \bar{y}(1,s)/\bar{M}(s)$ shown in Fig. 6 is nonminimum phase with an infinite number of zeros on the real axis of the s plane.

To help determine the locus of zeros of the transfer function in Eq. (6) as the location x of the sensor migrates between $x = 0$ at the left end of the beam to $x = 1$ at the right end, define a part of the transfer function as

$$g(x,\lambda) = \frac{1}{\lambda^3} \cdot \big[(\sinh\lambda \cdot \cos\lambda - \cosh\lambda \cdot \sin\lambda)(\cosh\lambda x$$

$$+ \cos\lambda x) + \sinh\lambda \cdot \sin\lambda(\sinh\lambda x + \sin\lambda x)$$

$$+ (1 - \cos\lambda \cdot \cosh\lambda)(\sinh\lambda x - \sin\lambda x)\big] \tag{12}$$

where

$$G(x,s) = g(x,\lambda)/2(1 - \cos\lambda \cdot \cosh\lambda)$$

Note that $g(x,\lambda)$ is defined with λ^3 in its denominator since the lowest-order term in the Taylor series of the numerator of $g(x,\lambda)$ contains the product λ^3. Hence, λ^3 is cancelled out, and the first term in the Taylor series of $g(x,\lambda)$ is made a constant. The significance of this will become clearer in Sec. III.

It is easy to verify that the function $g(x,\lambda)$ has the following properties:

$$g(x, -\lambda) = g(x,\lambda)$$

$$g(x, i\lambda) = g(x,\lambda)$$

where $i = \sqrt{-1}$. This means that the zeros of $g(x,\lambda)$ are symmetric with respect to the real axis, the imaginary axis, and the origin of the complex λ plane. All properties of the function $g(x,\lambda)$ for a force input case are true for an analogous function $h(x,\lambda)$ for a moment input case.

Solving the nonlinear complex equations in Eqs. (6) and (9) for the zeros by numerical methods is very difficult, but there is a subroutine available in the IMSL (International Mathematical and Statistical Libraries) package that can accomplish this task by employing Müller's method.

III. Locus of Zeros of a Beam on the *s* Plane and Its Analysis

Numerical Analysis of Zeros of a Beam

In this analysis, numerical solutions for $g(x,\lambda)$ and $h(x,\lambda)$ are obtained first. Then the relation $\lambda^4 = -s^2$ is used to construct the locus of zeros for the force input transfer function of a beam on the s plane from the pattern of zeros on the λ plane.

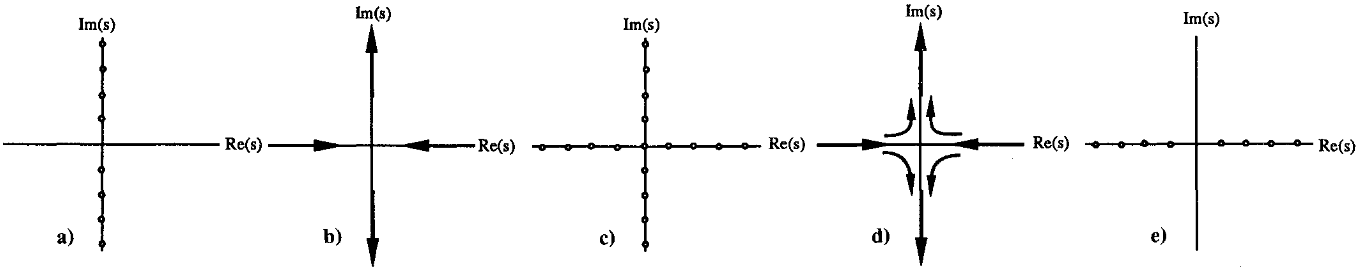

Fig. 7 Trends of zeros on the s plane as x increases for force and moment input cases ($x_p = 2/3$ for force input and $x_p = 1/2$ for moment input): a) $x = 0$, b) $0 < x < x_p$, c) $x = x_p$, d) $x_p < x < 1$, and e) $x = 1$.

Table 1 Zeros of $g(x,\lambda)$ around $x_p = 2/3$

	Zeros	
x Values	λ Plane	s Plane
$x = 0.66$	$\pm 1.429 \pm 1.429i$	± 4.084
$x = 2/3$	0	0
$x = 0.7$	$\pm 3.129, \quad \pm 3.129i$	$\pm 9.789i$

Table 2 Zeros of $h(x,\lambda)$ around $x_p = 1/2$

	Zeros	
x Values	λ Plane	s Plane
$x = 0.49$	$\pm 1.094 \pm 1.094i$	± 2.396
$x = 1/2$	0	0
$x = 0.51$	$\pm 1.551, \quad \pm 1.551i$	$\pm 2.406i$

There is a unique location of the translational displacement sensor on the beam at which the rigid-body modes are cancelled by the arrival of zeros at the origin of the s plane. Since $g(x,\lambda)$ is a complex nonlinear function, it does not appear possible to show analytically the explicit cancellation without the aid of a Taylor series.

Expansion of $g(x,\lambda)$ in a Taylor series (the symbolic manipulator software *Mathematica*™ written by S. Wolfram has been used) about $\lambda = 0$ with x as a parameter yields

$$g(x,\lambda) = \left(-\frac{4}{3} + 2x\right) + \left(\frac{2}{315} - \frac{2x}{90} + \frac{x^3}{18} - \frac{x^4}{18} + \frac{x^5}{60}\right)\lambda^4 + \left(-\frac{1}{311{,}850} + \frac{x}{56{,}700} - \frac{x^3}{7560} + \frac{x^4}{3780} - \frac{x^5}{5400} + \frac{x^7}{15{,}120} - \frac{x^8}{30{,}240} + \frac{x^9}{181{,}440}\right)\lambda^8 + \mathcal{O}(\lambda^{12}) \tag{13}$$

From Eq. (13) we can see that at $x = 2/3$ the constant term of $g(x,\lambda)$ becomes zero. This phenomenon can be demonstrated on both the λ and s planes. As x is increased, the zeros of $g(x,\lambda)$ on both axes of the λ plane move away from the origin, and the equivalent beam zeros on the s plane move away from the origin along the imaginary axis. Simultaneously, the zeros of $g(x,\lambda)$ on the $p = q$ and $p = -q$ lines of the λ plane approach the origin and map to beam zeros that approach the origin of the s plane along the real axis. The leading zeros of the approaching set reach the origins of the two planes where $x = 2/3$. Afterward, these zeros follow the departing set on the λ-plane axes and the imaginary axis of the s plane. See Fig. 7 and Table 1 for more details.

Unobservability of the Rigid-Body Mode

That point at $x = 2/3$ corresponds to the location at which the tangential component of the reaction acceleration is zero, as may be observed with the simplified rigid-beam configuration of Fig. 8. Since the angular acceleration caused by the force is $\alpha = FL/2I$, the tangential acceleration at a distance z from the center of mass is $a = \alpha \cdot z$, and the translational acceleration of the rigid beam in the opposite direction to a is $a_c = F/m$, where F is applied force at the left end of the beam, L length of the beam, m mass of the beam, and I moment of inertia of the beam about the center of mass ($mL^2/12$ for a homogeneous beam). The resultant acceleration $(FL/2I)z - F/m$ vanishes at $z = L/6$, and the distance from the left end is $x = z + 1/2 = 2/3$ for $L = 1$. This point is known as *a center of percussion*.[10] At such a point, the rigid-body mode cannot exist, which explains the cancellation of the poles and zeros at the s-plane origin.

Similarly, for the moment input transfer function, if a moment is applied and the translational displacement is measured as indicated in Fig. 9, then the rigid-body mode can be cancelled by zeros. Application of a moment to a rigid beam will cause the beam to rotate about its center of mass. Since the center of mass is located at $x = 1/2$ and $z = 0$, this point will not experience a reaction, i.e., a tangential component of acceleration. See Fig. 7 for more details. This pole-zero cancellation, as shown in Table 2, can be observed by expanding $h(x,\lambda)$ in a Taylor series about $\lambda = 0$ with x as a parameter.

$$h(x,\lambda) = (-2 + 4x) + \left(\frac{1}{45} - \frac{2x}{15} + \frac{x^2}{6} - \frac{x^4}{12} + \frac{x^5}{30}\right)\lambda^4 + \left(-\frac{1}{56{,}700} + \frac{x}{5670} - \frac{x^2}{2520} + \frac{x^4}{1080} - \frac{x^5}{900} + \frac{x^6}{2160} - \frac{x^8}{20{,}160} + \frac{x^9}{90{,}720}\right)\lambda^8 + \mathcal{O}(\lambda^{12})$$

When $x = 1/2$, if we place the sensor at the center of mass, the constant term is zero. Hence, the zeros arrive at the origin and cancel the rigid-body mode of the beam.

Contrary to general opinion,[8] when translational displacements are measured, rather than angular displacements, the zeros do not simply approach the origin from infinity along the real axis of the s plane as x is increased from 0 to 1. As described earlier, the first pair of zeros on the real axis actually pass through the origin, cancelling the rigid-body mode, as the sensor is moved past the center of percussion, and they subsequently depart to infinity along the imaginary axis of the s plane.

It is also interesting to note that what happens to the other zeros of the s plane apart from the first pair of zeros that appear on the real axis. If these other zeros reach the origin and cancel the rigid-body mode, this means there may exist another sensor location at which the rigid-body mode is cancelled. However, from the numerical results it can be easily observed that only the first pair of zeros on the real axis arrive at the origin and that all the other zeros on the real axis of the s plane do not. By implication the center of percussion is uniquely defined for the beam.

When an angular displacement sensor is used, unlike a translational sensor, pole-zero cancellation of the rigid-body translation mode does not occur. Rather, the angular rigid-body

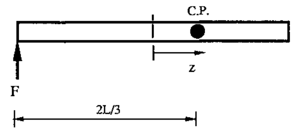

Fig. 8 Rigid beam with force input.

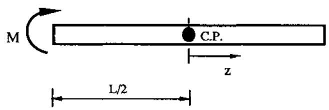

Fig. 9 Rigid beam with moment input.

mode $\theta(t)$ can be observed anywhere on the beam. These modes can be observed by examining the zeros of the associated force and moment output transfer functions. Since $\bar{\theta}(\lambda) = \partial \bar{y}(x,\lambda)/\partial x$, the equation of the zeros for $\bar{\theta}(s)/\bar{F}(s)$, or $g^{\theta}(x,\lambda)$, is obtained by taking the partial of $g(x,\lambda)$ with respect to x. For $\bar{\theta}(\lambda)/\bar{M}(s)$, the equation of the zeros, or $h^{\theta}(x,\lambda)$, is again obtained by taking the partial of $h(x,\lambda)$ with respect to x. The constant terms of the $g^{\theta}(x,\lambda)$ and $h^{\theta}(x,\lambda)$ expressions do not contain the parameter x and thus exclude any zeros from the origin of the associated λ and s planes. With no zeros at the origin to cancel the rigid-body mode, there can be no centers of percussion on the rigid beam for an angular displacement sensor.

IV. Conclusions

The true pattern of zeros for a beam, which is considered a basic component of a flexible structure, has been numerically investigated as a function of the sensor and actuator separation. Two cases were studied: one in which a force input is applied at the left end of the beam, and another in which the same beam end is subjected to a moment input. In both cases, translational displacement sensors were used. As the sensor is moved from the left end of the beam to the right end, the general pattern of zeros of the beam on the s plane changes as follows: Zeros move away from the origin along the imaginary axis and approach the origin along the real axis. When the sensor is located at a certain point, which has been shown to be unique and identified as a center of percussion, the first pair of zeros on the real axis arrive at the origin and cancel the rigid-body mode. The location of this point differs in the two cases: two-thirds of the span to the right for force input and at the center of the beam for moment input. Taylor series expansion of key transfer functions has been employed to explain the pole-zero cancellation. It has been further noted that the center of percussion is absent if angular displacement is measured.

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